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Mathematikaufgaben
> Analysis
> Mehrdimensionale Funktionen

Aufgabe: Gegeben ist die Funktion

$$f(x,y,z) = z^3 - 3z^2 + x^2 + 4xy + 2y + z + 20.$$

Bilde das 1., 2. und 3. Taylorpolynom zum Entwicklungsmittelpunkt (x,y,z) = (0, -1, 1).

Lösung: I. Zu stetig differenzierbaren mehrdimensionalen Funktionen f(x) wie Polynomen f: $\mathbf{R}^n$ -> **R** mit x = (x_1, … x_n) lassen sich Taylorpolynome T_k(x) um einen Entwicklungsmittelpunkt x_0 = (x_{01}, … x_{0n}) bilden von der Form:

$$T_k(x) = \sum_{|\alpha|=0}^{k} \frac{D^\alpha f(x_0)}{\alpha!}(x-x_0)^\alpha$$

mit: α = (α_1, … α_n), (Ordnung) |α| = α_1+…+α_n, α_i als natürliche Zahlen einschließlich 0, i=1, …n, (Falkutät) α! = α_1!·…·α_n! (mit: 0! = 1), (Potenz) x^α = $(x_1,...,x_n)^{(\alpha_1,...\alpha_n)} = x_1^{\alpha_1} \cdot ... x_n^{\alpha_n}$, (Ableitungsoperator) $D^\alpha = \dfrac{\partial^{\alpha_1}...\partial^{\alpha_n}}{\partial x_1^{\alpha_1}....\Im x_n^{\alpha_n}}$. Ein Taylorpolynom hat damit das Aussehen:

$$T_k(x) = \begin{cases} f(x_0) \\ +\dfrac{\partial f}{\partial x_1}(x_0)(x_1 - x_{01}) + ... + \dfrac{\partial f}{\partial x_n}(x_0)(x_n - x_{0n}) \\ +\dfrac{1}{2}\cdot\dfrac{\partial^2 f}{\partial x_1^2}(x_0)(x_1-x_{01})^2 + \dfrac{\partial^2 f}{\partial x_1 x_2}(x_0)(x_1-x_{01})(x_2-x_{02}) + ... + \dfrac{1}{2}\cdot\dfrac{\partial^2 f}{\partial x_n^2}(x_0)(x_n-x_{0n})^2 \\ + ... \end{cases}$$

II. Wir bilden zunächst die 1. bis 3. partiellen Ableitungen und haben unter Berücksichtigung des Entwicklungsmittelpunkts (x,y,z) = (0, -1, 1):

$f(x,y,z) = z^3 - 3z^2 + x^2 + 4xy + 2y + z + 20$ => f(0,-1,1) = 17

$\dfrac{\partial f(x,y,z)}{\partial x} = 2x + 4y$ => $D^{(1,0,0)} f(0,-1,1) = \dfrac{\partial f(0,-1,1)}{\partial x} = -4$

$\dfrac{\partial f(x,y,z)}{\partial y} = 4x + 2$ => $D^{(0,1,0)} f(0,-1,1) = \dfrac{\partial f(0,-1,1)}{\partial y} = 2$

$\dfrac{\partial f(x,y,z)}{\partial z} = 3z^2 - 6z + 1$ => $D^{(0,0,1)} f(0,-1,1) = \dfrac{\partial f(0,-1,1)}{\partial z} = -2$

$\dfrac{\partial^2 f(x,y,z)}{\partial x^2} = 2$ => $D^{(2,0,0)} f(0,-1,1) = \dfrac{\partial^2 f(0,-1,1)}{\partial x^2} = 2$

$\dfrac{\partial^2 f(x,y,z)}{\partial x \partial y} = \dfrac{\partial^2 f(x,y,z)}{\partial y \partial x} = 4$ => $D^{(1,1,0)} f(0,-1,1) = \dfrac{\partial^2 f(0,-1,1)}{\partial x \partial y} = 4$

$\dfrac{\partial^2 f(x,y,z)}{\partial x \partial z} = \dfrac{\partial^2 f(x,y,z)}{\partial z \partial x} = 0$ => $D^{(1,0,1)} f(0,-1,1) = \dfrac{\partial^2 f(0,-1,1)}{\partial x \partial z} = 0$

$$\frac{\partial^2 f(x,y,z)}{\partial y^2} = 0 \text{ => } D^{(0,2,0)} f(0,-1,1) = \frac{\partial^2 f(0,-1,1)}{\partial y^2} = 0$$

$$\frac{\partial^2 f(x,y,z)}{\partial y \partial z} = \frac{\partial^2 f(x,y,z)}{\partial z \partial y} = 0 \text{ => } D^{(0,1,1)} f(0,-1,1) = \frac{\partial^2 f(0,-1,1)}{\partial y \partial z} = 0$$

$$\frac{\partial^2 f(x,y,z)}{\partial z^2} = 6z - 6 \text{ => } D^{(0,0,2)} f(0,-1,1) = \frac{\partial^2 f(0,-1,1)}{\partial z^2} = 0$$

$$\frac{\partial^3 f(x,y,z)}{\partial x^3} = 0 \text{ => } D^{(3,0,0)} f(0,-1,1) = \frac{\partial^3 f(0,-1,1)}{\partial x^3} = 0$$

$$\frac{\partial^3 f(x,y,z)}{\partial x^2 \partial y} = \frac{\partial^3 f(x,y,z)}{\partial y \partial x^2} = 0 \text{ => } D^{(2,1,0)} f(0,-1,1) = \frac{\partial^3 f(0,-1,1)}{\partial x^2 \partial y} = 0$$

$$\frac{\partial^3 f(x,y,z)}{\partial x^2 \partial z} = \frac{\partial^3 f(x,y,z)}{\partial z \partial x^2} = 0 \text{ => } D^{(2,0,1)} f(0,-1,1) = \frac{\partial^3 f(0,-1,1)}{\partial x^2 \partial z} = 0$$

$$\frac{\partial^3 f(x,y,z)}{\partial y^3} = 0 \text{ => } D^{(0,3,0)} f(0,-1,1) = \frac{\partial^3 f(0,-1,1)}{\partial y^3} = 0$$

$$\frac{\partial^3 f(x,y,z)}{\partial y^2 \partial x} = \frac{\partial^3 f(x,y,z)}{\partial x \partial y^2} = 0 \text{ => } D^{(1,2,0)} f(0,-1,1) = \frac{\partial^3 f(0,-1,1)}{\partial x \partial y^2} = 0$$

$$\frac{\partial^3 f(x,y,z)}{\partial y^2 \partial z} = \frac{\partial^3 f(x,y,z)}{\partial z \partial y^2} = 0 \text{ => } D^{(0,2,1)} f(0,-1,1) = \frac{\partial^3 f(0,-1,1)}{\partial x \partial y^2} = 0$$

$$\frac{\partial^3 f(x,y,z)}{\partial z^3} = 6 \text{ => } D^{(0,0,3)} f(0,-1,1) = \frac{\partial^3 f(0,-1,1)}{\partial z^3} = 6$$

$$\frac{\partial^3 f(x,y,z)}{\partial z^2 \partial x} = \frac{\partial^3 f(x,y,z)}{\partial x \partial z^2} = 0 \text{ => } D^{(1,0,2)} f(0,-1,1) = \frac{\partial^3 f(0,-1,1)}{\partial x \partial z^2} = 0$$

$$\frac{\partial^3 f(x,y,z)}{\partial z^2 \partial y} = \frac{\partial^3 f(x,y,z)}{\partial y \partial z^2} = 0 \text{ => } D^{(0,1,2)} f(0,-1,1) = \frac{\partial^3 f(0,-1,1)}{\partial y \partial z^2} = 0$$

III. Das 1. bis 3. <u>Taylorpolynom</u> lautet nun:

T_1(x,y,z) = 17 – 4(x-0) + 2(y+1) – 2(z-1) = 17 – 4x + 2(y+1) – 2(z-1)

T_2(x,y,z) = 17 – 4x + 2(y+1) – 2(z-1)

$$+ \frac{1}{2}\cdot 2(x-0)^2 + 4(x-0)(y+1) + 0(x-0)(z-1) + \frac{1}{2}\cdot 0(y+1)^2 + 0(y+1)(z-1) + \frac{1}{2}\cdot 0(z-1)^2$$

$$= 17 - 4x + 2(y+1) - 2(z-1) + x^2$$

T_3(x,y,z) = 17 – 4x + 2(y+1) – 2(z-1) + x^2 + $\frac{1}{6}\cdot 0(x-0)^3 + \ldots + \frac{1}{6}\cdot 6(z-1)^3$

$$= 17 - 4x + 2(y+1) - 2(z-1) + x^2 + 4x(y+1) + (z-1)^3$$

Es gilt:

T_3(x,y,z) = f(x,y,z)

auf Grund der Tatsache, dass die Funktion $f(x,y,z) = z^3 - 3z^2 + x^2 + 4xy + 2y + z + 20$ wie das Taylorpolynom T_3(x,y,z) ein Polynom 3. Grades ist.

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